Gamma discounting and expected net future value

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Abstract

Recent research suggests that the long term future should be discounted with a declining discount rate. One such line of research, exemplified by Weitzman [11], shows that the certainty equivalent discount rate is declining when future capital productivity is uncertain. However, in a recent paper Gollier [4] puts forward a puzzle that casts doubt on the validity of this conclusion. He asserts that using expected net future value, rather than conventional expected net present value, implies that the certainty equivalent discount rate *increases* over time. This paper resolves the apparent puzzle by encompassing the models of Gollier [4] and Weitzman [11]. In fact, Gollier [4] proves that as the *evaluation date* moves further into the future, the discount rate at a given point in time will increase. However, given a particular evaluation date, the schedule of discount rates is declining.

1 Introduction

One of the most powerful arguments for declining social discount rates is that uncertainty (and persistence) in the discount rate implies that the certainty-equivalent discount rate is declining. Weitzman [10, 11] provides an extremely simple account of the argument. Newell and Pizer [8] and Groom et al. [6] develop more sophisticated econometric analyses which, nevertheless, have the same conceptual underpinning. The case for declining discount rates based upon uncertainty has recently been accepted by the UK Government and is now incorporated into the official advice in HM Treasury [7].

In a recent paper, however, Gollier [4] puts forward a puzzle which casts doubt on the validity of the case based upon uncertainty. The puzzle, which is explained more fully in section 2, is based on a simple thought experiment. Compare the following two alternatives: (1) invest £1 in a project with a certain 3% annual rate of return, yielding a certain payoff of £400 in 200 years; and (2) invest £1 on capital markets that are equally likely to return 0% or 5%, yielding a payoff of either £1 or £22,000 in 200 years.

Faced with this choice, it might seem obvious that a risk neutral investor would prefer the investment on the capital markets, with an expected payoff of £11,000. Indeed, this is optimal according to the *expected net future value* (ENFV) criterion, which compares costs and benefits in their equivalent future values in 200 years. However, using the conventional *expected net present value* (ENPV) — where costs and benefits are discounted back to their present values — the result is reversed and the certain project is preferred. With the capital markets as numeraire, the ENPV

of investing on the markets is zero, while the ENPV of investing in the safe project is approximately £200.[1] The two criteria give opposite results for this uncertain investment decision. In contrast, ENFV and ENPV give identical results under perfect certainty — uncertainty appears to force them to part company.

Following on from this thought experiment, Gollier [4] appears to demonstrate that under the *expected net future value* criterion, the certainty-equivalent discount rate is *increasing* with time. This is obviously incompatible with the broad conclusion in the literature. Referring to the results presented by Weitzman [10, 11], Gollier states that 'we cannot both be right. In fact, to tell the truth, I believe that we are both wrong'.

This paper shows that the 'puzzle' put forward by Gollier has a straightforward resolution and that, in essence, Weitzman and Gollier *can* both be right. We demonstrate that the model in Gollier [4] does not prove that the discount rate increases with the passage of time. On the contrary, the socially efficient discount rate declines à la Weitzman [10, 11] irrespective of the criteria employed in CBA. However, we demonstrate that there is a sense in which Gollier is correct in saying that the discount rate is increasing with time. As the *evaluation date*, that is the numeraire date employed for assessing the investment, moves further into the future, the discount rate at a particular point in time increases. In this sense, Gollier and Weitzman are both right.

The paper proceeds as follows. In section 2 we describe the puzzle proposed by

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[1]The ENPV of investing on the markets is $-£1 + 0.5(£1 \cdot e^{-0\% \times 200}) + 0.5(£22,000 \cdot e^{-5\% \times 200}) = £0$. In contrast, the ENPV of the certain project is $-£1 + 0.5(£400 \cdot e^{-0\% \times 200}) + 0.5(£400 \cdot e^{-5\% \times 200}) \approx £200$.

4

Gollier [4] and present his argument that the certainty-equivalent discount rate increases under ENFV. In section 3 we discuss his explanation for the puzzle — the two decision criteria imply different intertemporal allocation of risk. We show, however, that this explanation is incorrect. In section 4, we show that the puzzle arises from confusion about the role of the *evaluation date*, and we present our solution in section 5. In section 6, we move beyond this debate and note that under uncertainty, whether or not a project is efficient depends upon the evaluation date. We develop some simple 'intergenerational efficiency rules' to determine how far into the future a given project would pass *retrospective* cost benefit analysis. We conclude in section 7.

2   The puzzle

The puzzle described by Gollier [4] is based upon an investment of one unit of consumption at $t = 0$ which yields a certain payoff of Z units at time $t = T$. The (social) opportunity cost of this investment is given by the random variable $\tilde{r}$ drawn from $[\underline{r}, \bar{r}]$ with cumulative distribution function $G(\tilde{r})$.[2]   In general, we define a

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[2]This formulation, employed by Gollier [4] and Weitzman [10, 11], is a rather unusual and unrealistic specification of uncertainty. It implicitly assumes an extremely high degree of *persistence* in the discount rate. (That is, once the uncertainty is resolved, the discount rate remains constant from then on.) This is an approximation to more realistic econometric models such as those of Newell and Pizer [8] and Groom et al. [6]. As these authors find US and UK discount rates to show moderate persistence, the approximation — while unrealistic — is not inappropriate for our purposes.

Gollier project to have a flow of net benefits given by:

$$B(t) = -\delta(0) + Z\delta(T) \tag{1}$$

where $\delta(t)$ represents the Dirac delta function, which has unit area at t and is zero elsewhere.

2.1 Expected net present value

The ENPV of an investment is found by converting all cash flows into common units at $t = 0$. Net benefits $B(t)$ accruing at time t are multiplied by the certainty-equivalent discount function $D_c(t) = \mathrm{E}e^{-\tilde{r}t}$ and summed to yield:

$$\mathrm{ENPV} = \int_0^T B(t)\mathrm{E}e^{-\tilde{r}t}dt = \int_{\underline{r}}^{\bar{r}} \int_0^T B(t)e^{-\tilde{r}t}dt \ dG(\tilde{r}) \tag{2}$$

Substituting the net benefit function from equation (1) to equation (2) reveals that the expected net present value of a Gollier project is:

$$\mathrm{ENPV} = Z\mathrm{E}e^{-\tilde{r}T} - 1 \tag{3}$$

The certainty-equivalent average[3] discount rate, $r_{ca}(t)$ is the discount rate which, when applied over $[0, t]$, yields the certainty-equivalent discount function: $e^{-r_{ca}t} = D_c(t) = \mathrm{E}e^{-\tilde{r}t}$. In other words, the certainty-equivalent average discount rate is defined by:

$$r_{ca}(t) = -\frac{1}{t}\ln \mathrm{E}e^{-\tilde{r}t} \tag{4}$$

It is straightforward to prove that $r_{ca}(t)$ is a declining function of time t.[4]

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[3]Note that the certainty-equivalent *average* discount rate is not generally equal to the certainty-equivalent *marginal* discount rate, except in the case when these discount rates are constant.

[4]For instance, see the more general proposition 6 in the Appendix and set $s = \tau = 0$.

2.2 Expected net future value

The ENFV of an investment is found by converting all cash flows into common units at a future point in time, generally the project end date, $t = T$. To do this, net benefits accruing at time t are multiplied by certainty-equivalent discount function $D_c(t, T) = \mathrm{E}e^{-\tilde{r}(t-T)}$ and summed to yield:

$$\mathrm{ENFV} = \int_0^T B(t)\mathrm{E}e^{-\tilde{r}(t-T)}dt = \int_{\underline{r}}^{\bar{r}} \int_0^T B(t)e^{-\tilde{r}(t-T)}dt \; dG(\tilde{r}) \tag{5}$$

Substituting the net benefit function from equation (1) to equation (5) shows that the ENFV of a Gollier project is:

$$\mathrm{ENFV} = Z - \mathrm{E}e^{\tilde{r}T} \tag{6}$$

Rewriting this for convenience, we see that ENFV will be positive (and the investment worthwhile) provided that $Z\left(\mathrm{E}e^{\tilde{r}T}\right)^{-1} - 1 > 0$. Gollier compares this equation with equation (3) and notes that the corresponding certainty-equivalent return under ENFV is given by the equation $e^{R(t)} = \mathrm{E}e^{\tilde{r}t}$, from which he infers that the certainty-equivalent discount rate is:

$$R(t) = \frac{1}{t}\ln\left(\mathrm{E}e^{\tilde{r}t}\right) \tag{7}$$

It is straightforward to show that $R(t)$ is increasing over time.[5] A puzzle therefore appears to arise. Under ENPV, the certainty-equivalent schedule of discount rates is declining, yet under ENFV, the certainty-equivalent schedule of discount rates appears to be increasing. Gollier [4] proposes a solution to the puzzle which we now consider.

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[5]See proposition 2 by Gollier [4]. Proposition 7 in the Appendix proves a corresponding result in a more general setting.

3 Gollier's explanation: risk allocation

Gollier [4] argues that the puzzle is solved by recognising that ENFV and ENPV imply different (and arbitrary) intertemporal allocations of risk:

> *Taking the expected net future value is equivalent to assuming that all risks will be borne by the future generation...Using the expected net present value implicitly means it is the current generation who bears the risk...Because the two approaches lead to radically different recommendations, we see that, to solve the problem, we cannot escape the discussion of who should bear which risk.*

In other words, Gollier [4] asserts that an explicit treatment of optimal intertemporal risk allocation, such as developed by Gollier [2, 3, 5], is necessary before any sensible conclusions can be drawn. However, it seems odd that different risk allocations could alter the optimal decision of a risk-neutral investor. Indeed, the statement that ENFV allocates risk to future generations is incorrect. ENFV and ENPV are merely decision criteria. Both criteria can be applied in situations where the risk is borne by either present or future generations.

To see this, consider the following two thought experiments. The first thought experiment was discussed above — an investor chooses between investing £1 for T years in either: (1) a safe deposit with a certain return r, yielding a certain payoff of e^{rT}; or (2) the market, with a stochastic return of $\tilde{r}$, with expected payoff of $Ee^{\tilde{r}T}$. Recall that a risk-neutral investor applying the ENPV criterion would invest in the deposit. The same risk-neutral investor applying the ENFV criterion would choose to invest on the market. The risk here is borne by the future generation.

In contrast, now consider a second situation where the risk is borne by the present generation. Suppose you need a certain payoff of $Z = e^{rT}$ at time T, and can choose between two options to achieve this: (1) invest £1 in a safe deposit, yielding the certain payoff of $Z = e^{rT}$; or (2) purchase a bond on market which pays £Z for certain on maturity in T years. The current price of the bond depends upon the market interest rate, $\tilde{r}$, and we suppose the bond is purchased before this interest rate is known. The expected bond price is E$Ze^{-\tilde{r}T}$. With the same numbers as above, the bond price is equally likely to be £400 or £0.02, depending upon the prevailing interest rate, with an expected price of £200. As before, a risk-neutral investor applying the ENPV criterion would invest in the safe deposit. And, as before, the same risk-neutral investor using ENFV would (weakly) prefer to invest on the market and buy the bond.

[Insert Table I about here.]

These two thought experiments, summarised in table I, are analogues of one another. The first one imposed risk upon the future generation, the second one imposed risk upon the present generation. Both were able to be evaluated with the ENPV and ENFV criteria. The risk allocation did not change the result — the choice of criterion was the critical factor. ENPV favours more secure investments while ENFV favours higher risk investments, *no matter which generation bears the risk*. The puzzle is not solved by attributing the difference between the ENPV and ENFV criteria to differential intertemporal risk allocation. As we shall see, considering the expected net value at a general *evaluation time τ* provides some more insight.
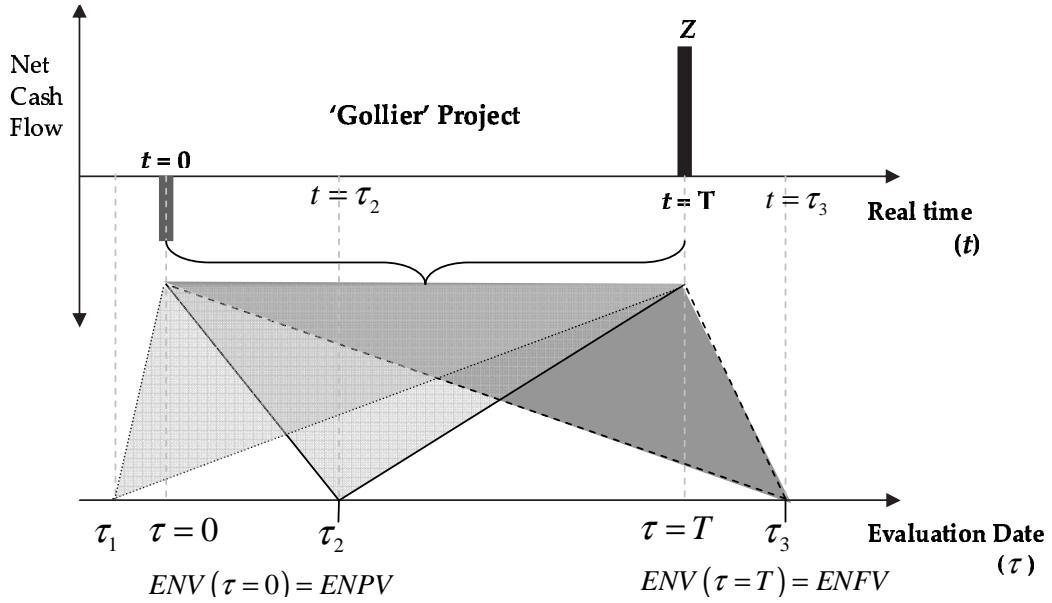
9

Figure 1: A Gollier project in [0, T] can be evaluated at any date τ.

4   An alternative explanation: evaluation date

Up until this point, we have considered either ENPV (evaluation time $\tau = 0$), or ENFV (evaluation time $\tau = T$). It is helpful to generalise this analysis and to consider cost benefit analysis at a general evaluation time τ. Let us examine a project, such as a Gollier project in figure 1, occurring over a given interval in real time, $[0, T]$. The entire project (not only the remaining portion) can be evaluated from the perspective of any date, τ, prospectively (e.g. at τ_1), concurrently (e.g. at τ_2) or retrospectively (e.g. at τ_3).[6] In a retrospective evaluation, $(\tau \geq T)$, the planner evaluates the project based on the information available *ex ante* — as if the uncertainty in the discount rate has not yet been resolved. This is equivalent to asking: 'what would a future generation want us to do, given the information

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[6]Note that moving the evaluation date forwards in time is equivalent to moving the stream of net benefits backwards in time. This is reflected in the fact that equation (8) depends only upon $(t - \tau)$.

available to us now.' Denote the expected net value at evaluation date τ to be $\mathrm{ENV}(\tau)$. Then, analogously to the definition for ENPV in equation (2), we have:

$$\mathrm{ENV}(\tau) = \mathrm{E} \int_0^T B(t) e^{-\tilde{r}(t-\tau)} dt \tag{8}$$

For a project evaluated at time τ, the certainty-equivalent discount function is given by:

$$D_c(t, \tau) = e^{-r_{ca}(t,\tau)(t-\tau)} = \mathrm{E} e^{-\tilde{r}(t-\tau)} \tag{9}$$

This discount function is normalised so that consumption at the evaluation date, $t = \tau$, has a weight of unity.[7] Given the discount function in equation (9), the corresponding certainty-equivalent average discount rate is given by:

$$r_{ca}(t, \tau) = -\frac{1}{t-\tau} \ln \mathrm{E} e^{-\tilde{r}(t-\tau)} \tag{10}$$

PROPOSITION 1. *The certainty-equivalent average discount rate, $r_{ca}(t,\tau)$, declines with the passage of time to $\lim_{t\to\infty} r_{ca} = \underline{r}$, but it is increasing in the evaluation date, τ.*

Proof. Showing $\partial r_{ca}(t,\tau)/\partial t \leq 0$ is straightforward, as noted above. Proof that $\partial r_{ca}(t,\tau)/\partial \tau \geq 0$ follows from the fact that $\partial r_{ca}/\partial t = -\partial r_{ca}/\partial \tau$. ∎

Proposition 1 suggests a solution to the puzzle. Although the certainty-equivalent discount rate is declining with t, it is increasing in τ, the evaluation time. It turns out that the puzzle arises from confusing these two time variables. Reconsider Gollier's ENFV certainty-equivalent average discount rate in equation (7). According to the generalised expression in equation (10), if the evaluation time is $\tau = T$, cash

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[7] Other normalisations are valid, of course, but the results that follow do not depend upon the particular normalisation used, as we prove in propositions 6 and 7 in the Appendix. We are grateful to Christian Gollier for alerting us to this point.

11

flows accruing at t must be multiplied by the discount function $e^{-\tilde{r}(t-T)}$, and the certainty-equivalent average discount rate is given by:

$$r_{ca}(t, T) = -\frac{1}{t - T} \ln \mathrm{E} e^{-\tilde{r}(t-T)} \qquad (11)$$

Now equations (7) and (11) should be identical, but inspection shows that Gollier's specification of $R(t)$ under ENFV is incomplete. Now note that at $t = 0$, the certainty-equivalent average discount rate in equation (11) is given by:

$$r_{ca}(0, T) = \frac{1}{T} \ln \left(\mathrm{E} e^{\tilde{r}T} \right) \qquad (12)$$

which is almost identical to equation (7). The only difference is that the evaluation date, $\tau = T$, in equation (12), has been replaced by the passage of time, t, in equation (7).

So although Gollier correctly proved that the certainty equivalent discount rate is increasing in one time variable, this variable is the *evaluation date*, τ, rather than the passage of time, t. The certainty equivalent discount rate is declining with the passage of time, as per Weitzman [10]. In this sense, both Weitzman and Gollier are right. Figure 2 provides a simple illustration, assuming that the discount rate is drawn from a gamma distribution[8] with mean 4% and standard deviation 1%. For a given evaluation date τ, the certainty-equivalent average discount rate *declines* as time t increases. However, it is shifted up when the evaluation date τ is moved further into the future.[9]

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[8]The density function of the gamma distribution is given by: $g(r) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}$ with corresponding mean $\mu = \alpha/\beta$ and variance $\sigma^2 = \alpha/\beta^2$.

[9]Equivalently, the discount rate at a particular time horizon, T, *declines* as the time horizon T increases, but is shifted up when the evaluation date τ increases.
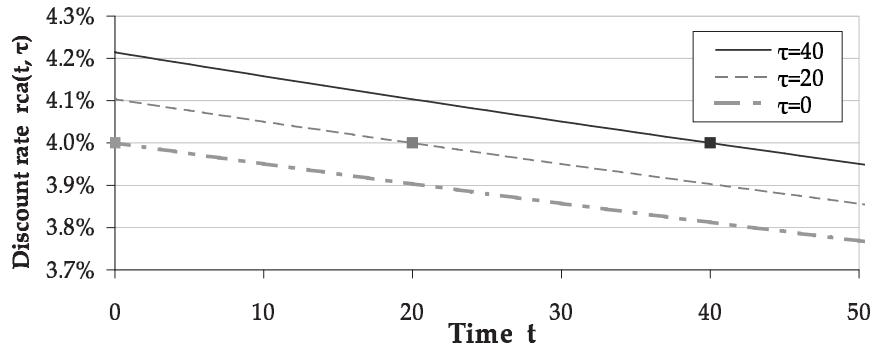
Figure 2: Certainty-equivalent average discount rates for three evaluation dates.

5 Solving the puzzle

The foregoing analysis resolves the confusion between the passage of time and the evaluation date, but does not completely solve the puzzle that, under uncertainty, a project that passes a cost benefit analysis using the ENPV criterion may fail using the ENFV criterion. As ENPV and ENFV represent evaluations at different dates, the final piece of the puzzle is to explain why, under uncertainty, a project can be judged efficient at one evaluation date, and inefficient at another.

A good starting point is Strotz [9], who recognised that declining utility discount rates can generate time-inconsistent planning — where the optimum path as determined at one point in time is different to that determined at a later date, for no reason other than the passage of time. Strotz [9] asked:

> If [an agent] is free to reconsider his plan at later dates, will he abide by
> it or disobey it—even though his original expectations of future desires
> and means of consumption are verified?'

Although the underlying phenomenon here is different — we have declining con-

13

sumption, not utility, discount rates — the result is the same.[10] Because the discount rate is uncertain, the certainty-equivalent discount rate is declining through time, and the discount rate applied at a particular point in time depends upon the evaluation date. The result is that a project that passes a cost benefit analysis from the viewpoint of a planner at time τ_1 may fail the test when examined by a planner at time τ_2.[11] Viewed through this lens, the puzzle posed by Gollier [4] is not really a puzzle at all — under uncertainty, ENPV and the ENFV criteria can recommend different courses of action because certainty-equivalent discount rates are declining.

And yet, for a policymaker, the foregoing analysis raises more questions than it answers. If a project designed for the benefit of future generations passes a cost benefit analysis now, but would fail the same test using a later evaluation date, should we invest in it? Or, more generally, how should we make decisions when there is a conflict between analyses using different evaluation dates? Whose date

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[10]If the underlying utility discount rate is constant, a varying consumption discount will not produce time-inconsistent planning, as Dasgupta et al. [1] note. Neither Weitzman [11] nor Gollier [4] analyse an optimal growth model, so there is no explicit utility discount rate here.

[11]There are two reasons to refrain from describing this intertemporal conflict as 'time inconsistency'. First, time inconsistency normally involves a planner at a future date finding it optimal to deviate from an earlier plan. Although this is possible here (e.g. planners at $\tau_1 < 0$ and $\tau_2 = 0$ might disagree about the optimal action at $t = 0$), we are more interested in the case where a future planner (at $\tau_3 > T$) *retrospectively* disapproves of the action taken at $t = 0$ by the planner at $\tau = 0$. Second, Newell and Pizer [8] note that when the discount rate decline is generated by 'dynamic uncertainty about future events', good decisions *ex ante* may be regrettable once information is revealed *ex post*. Here, the *ex ante* evaluation assumes that the discount rate uncertainty will be resolved once and for all *ex post* (see footnote 2). However, the *ex post* evaluation is conducted based on the same (unresolved) discount rate uncertainty. This amounts to revealing that the discount rate uncertainty has not been resolved as anticipated *ex ante*.

should be employed? These questions are difficult, and we do not pretend to have answers for them. Nonetheless, this next section proposes some guidance on the degree of conflict between the vantage points of different generations.

6   Project efficiency and the evaluation date

To place some formal structure on these questions, we define the *internal rate of return* of a project as the constant, certain discount rate that yields a net value $NV(\tau, r)$ equal to zero for that project. The internal rate of return of a project does not change with the evaluation date. Formally,

DEFINITION 1. *The* internal rate of return *of a project is* $r^* = \{r : NV(\tau, r) = 0\}$. *For a Gollier project,* $r^* = \frac{1}{T} \ln Z$.

As the efficiency of a project is assessed differently from different evaluation dates, the following definition is useful.

DEFINITION 2. *A project is* τ-efficient *if passes a cost benefit analysis at evaluation date* τ.

Employing these definitions, we can prove the following.

PROPOSITION 2. *For the set of projects with* $\mathrm{E}\tilde{r} > 0$ *and* $cov(\tilde{r}, NV(\tau, \tilde{r})) < 0$, *if a project is* τ'-efficient, *then it is also* τ-efficient *for all* $\tau \leq \tau'$.

Proof.   Differentiating equation (8) with respect to the evaluation date, we see

15

that:

$$\frac{d\text{ENV}(\tau)}{d\tau} = \text{E}\tilde{r}\int_0^T B(t)e^{-\tilde{r}(t-\tau)}dt \tag{13}$$

$$= \text{E}\left[\tilde{r}.NV(\tau,\tilde{r})\right] \tag{14}$$

$$= \text{E}\tilde{r}.\text{ENV}(\tau,\tilde{r}) + cov(\tilde{r}, NV(\tau,\tilde{r})) \tag{15}$$

By assumption $\text{E}\tilde{r} > 0$ and $cov(\tilde{r}, NV(\tau,\tilde{r})) < 0$ so from equation (15) it follows that $\text{ENV}(\tau,\tilde{r}) \leq 0 \Rightarrow d\text{ENV}(\tau)/d\tau < 0$. As such, if $\text{ENV}(\tau',\tilde{r}) \leq 0$, then $\text{ENV}(\tau,\tilde{r}) < 0$ for all $\tau > \tau'$. In other words, if ENV is non-positive at a given evaluation date, it must be negative for all later evaluation dates. It follows directly from this that if ENV is positive at a given date, it cannot have been negative at an earlier date. That is, if $\text{ENV}(\tau',\tilde{r}) > 0$, then $\text{ENV}(\tau,\tilde{r}) > 0$ for all $\tau \leq \tau'$, and thus the project is τ-efficient for all $\tau \leq \tau'$. ∎

The assumption that $\text{E}\tilde{r} > 0$ amounts to requiring that the expected opportunity cost of the investment is positive. This is likely to be true. Note that the support of $\tilde{r}$ is unrestricted — proposition 2 holds even if the support of $\tilde{r}$ includes negative values, provided $\text{E}\tilde{r} > 0$. The assumption that $cov(\tilde{r}, NV(\tau,\tilde{r})) < 0$ requires that a higher discount rate produce a lower net value of the investment. Many investments satisfy this requirement because an 'investment', by definition, involves costs now in return for benefits in the future.[12] However, some investments also have costs which accrue after a long stream of benefits, such as nuclear power plants. The net value of such projects may increase in r over some range and hence $cov\left(\tilde{r}, NV\left(\tau,\tilde{r}\right)\right)$ may be positive.[13]

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[12]Note that $cov(\tilde{r}, NV(\tau,\tilde{r}))$ is a function of τ, and for some investments $cov(\tilde{r}, NV(\tau,\tilde{r})) > 0$ for $\tau > T$.

[13]In these cases, the internal rate of return may not be unique. Despite this possibility, propo-

Proposition 2 implies that if a (risk-neutral) future generation finds a safe investment to be efficient relative to the risky market alternative, then the (risk-neutral) present generation will also judge the safe investment efficient. The converse, however, is not true: even if the present generation finds the safe investment to be efficient, the future generations may judge the risky investment to be efficient. Indeed, this case describes the original thought experiment, in table I, where a certain investment was preferred under the ENPV criterion and the risky market investment was preferred under the ENFV criterion.

What is the reason for this result? Mathematically, it is because the discount rate at any given time, t, is lower the earlier the evaluation date τ. As such, for most investments, where the costs accrue before the benefits, a lower discount rate implies a higher ENV. As such, the earlier the evaluation date, the more attractive the investment.

We have seen that if a project is efficient at a future evaluation date, it is also efficient today. It is convenient, therefore, to define the *critical evaluation date* $\bar{\tau}$ as the evaluation date before which the project is assessed to be efficient, and after which it is assessed to be inefficient. By proposition 2, a project is efficient for all evaluation dates $\tau < \bar{\tau}$ where $\bar{\tau} = \{\tau : \text{ENV}(\tau) = 0\}$.

We can now proceed to investigate the circumstances in which different generations would approve, or disapprove, of investment in a project. In particular, we aim to determine when a project will be judged to be: (1) *unanimously efficient*, meaning

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sition 2 holds for a larger set of projects than those with a unique internal rate of return and $NV(\tau, r)$ monotonically decreasing in r. This is because $cov(r, NV(\tau, r))$ may be negative even if $NV(\tau, r)$ is increasing in r over a subset of the domain.

that the project is considered to be efficient no matter what the evaluation dates ($\bar{\tau}$ is unbounded); (2) *time horizon efficient*, meaning efficient for evaluation dates up to receipt of the payoff at time T, ($\bar{\tau} > T$); (3) *currently efficient*, meaning efficient at the current date, ($\bar{\tau} > 0$); and (4) *never efficient* at any evaluation date, including dates in the past, ($\bar{\tau} \to -\infty$).

With a certain discount rate, definition 1 implies that the $\text{ENV}(\tau)$ of any flow of net benefits $B(t)$ is isomorphic to a flow of benefits given by:

$$B(t) = -\delta(0) + e^{r^*T}\delta(T) \tag{16}$$

where δ is the Dirac delta function and r^* is the internal rate of return. While this does not hold under uncertainty, we can nevertheless gain some useful insight by studying the set of Gollier projects, where the payoff $Z = e^{r^*T}$. For such a Gollier project, equation (8) becomes:

$$\text{ENV}(\tau) = \text{E}\left[e^{\tilde{r}\tau}\left(e^{(r^* - \tilde{r})T} - 1\right)\right] \tag{17}$$

PROPOSITION 3. **(Intergenerational efficiency rules)**

A Gollier project with internal rate of return r^* is said to be:

$$\left\{\begin{array}{l} \textit{unanimously efficient} \\ \textit{time horizon efficient} \\ \textit{currently efficient} \\ \textit{never efficient} \end{array}\right. \textit{for} \left\{\begin{array}{l} r^* > \bar{r} \\ r^* > r_{ca}(0, T) \\ r^* > r_{ca}(T, 0) \\ r^* < \underline{r} \end{array}\right.$$

Proof. If $r^* > \bar{r}$ then $e^{(r^* - \tilde{r})T} > 1$ and as $\text{E}e^{\tilde{r}\tau} > 0$ it follows from equation (17) that $\text{ENV}(\tau) > 0$ for all τ and the project is unanimously efficient. At the other extreme, if $r^* < \underline{r}$, then $e^{(r^* - \tilde{r})T} < 1$ and $\text{ENV}(\tau) < 0$ for all τ and the project is never efficient.

For the other two cases, a Gollier project is τ-efficient if $Z > \bar{Z}$, where $\bar{Z}$ is the payoff that implies $\mathrm{ENV}(\tau) = 0$. From equation (17), that is:

$$\bar{Z}(\tau) = \frac{\mathrm{E}e^{\tilde{r}\tau}}{\mathrm{E}e^{-\tilde{r}(T-\tau)}} \tag{18}$$

For the project to be time horizon efficient — that is, to be T-efficient — equation (18) implies that the payoff Z must exceed $\bar{Z}(T) = \mathrm{E}e^{\tilde{r}T}$. The internal rate of return for a marginal T-efficient project is given by $1/T \ln \bar{Z} = 1/T \ln \mathrm{E}e^{\tilde{r}T} = r_{ca}(0, T)$. Thus any project with an internal rate of return $r^* > r_{ca}(0, T)$ is time horizon efficient.

For a project to be currently efficient requires $Z > \bar{Z}(0) = 1/\mathrm{E}e^{-\tilde{r}T}$. The internal rate of return for the marginal project is $-1/T \ln \mathrm{E}e^{-\tilde{r}T} = r_{ca}(T, 0)$. Thus a project with internal rate of return $r^* \geq r_{ca}(T, 0)$ is currently efficient. ∎

In other words, if the internal rate of return of the project exceeds the upper bound of the support for the discount rate, we observe a unanimous judgement of efficiency from all generations. This is entirely intuitive — if the certain project has a payoff exceeding the best possible outcome on the markets, all generations will support the investment. The other extreme case is similarly intuitive: no generation will judge a project to be efficient which has an internal rate of return below the lower bound of the support of the discount rate.

The intergenerational efficiency rules in proposition 3 represent a first step towards dealing with the intergenerational conflict arising from declining discount rates. Clearly, policymakers can invest with confidence in a project that is unanimously efficient, and they would probably be comfortable investing in projects that are time-horizon efficient. For short-term projects, the simple requirement of current

efficiency might be enough. With the recent policy focus on ensuring 'sustainability', however, perhaps this sets the bar too low, but ultimately, this is for governments to decide.

7 Conclusion

This paper solves an intriguing puzzle, discovered by Gollier [4], that certainty-equivalent discount rates decline over time when the expected net present value (ENPV) criterion is employed, but appear to *increase* with time when the logically equivalent expected net future value (ENFV) criterion is used. The puzzle is not a mere academic *curiosum* — given that at least one national government has already adopted declining discount rates based upon the uncertainty rationale in Weitzman [10, 11], resolving the puzzle is crucial. Gollier [4] lays down the gauntlet with his conclusion that 'we cannot both be right. In fact, to tell the truth, I believe that we are both wrong'.

Gollier's solution to the puzzle centers upon the idea that the ENPV criterion imposes the investment risk on the present, while the ENFV criterion imposes the risk on the future. However, this paper shows that the allocation of risk has nothing to do with the appraisal criterion. In fact, the choice of ENPV or ENFV simply represents the choice of a specific *evaluation date*. Thus Gollier's explanation of the puzzle is not completely satisfactory.

Our conclusion, perhaps surprisingly, is that Weitzman and Gollier *are* both right. We show that the schedule of certainty-equivalent discount rates declines with the passage of time, but increases with the evaluation date. It is in this sense that

Gollier [4] is correct to say that the discount rate increases with time. Proponents of declining discount rates and institutions employing them — such as the UK Government — can take solace from this paper that their approach has not been invalidated.

Our resolution of the puzzle exposes the problem that investment choice is awkward in the Weitzman [10, 11] setting because the optimal investment depends upon the evaluation date. Indeed, the main thrust of Gollier [4] — that 'our criteria are arbitrary' — is correct. A fuller analysis is required. One such line of investigation by Gollier [2, 3, 5] assumes a constant utility discount rate and considers the effect of different risk preferences on the yield curve. An alternative approach, discussed in this paper, is to accept the presence of intergenerational conflict and to attempt to formulate a set of rules providing guidance. Both approaches have some merit. Indeed, in the murky waters of intergenerational policy, any theoretical advance providing a ray or two of light is to be welcomed.

Appendix

This appendix shows that proposition 1 holds irrespective of the time at which the discount function is normalised to unity, s. To see this, suppose we modify equation (8) to read:

$$\text{ENV}(\tau) = k\text{E} \int_0^T B(t) e^{-\tilde{r}(t-\tau)} dt \qquad (19)$$

where k is a normalisation constant employed to ensure the discount function at a particular time $t = s$ is unity. In other words, we require

$$k = \frac{1}{\text{E}e^{-\tilde{r}(s-\tau)}} \qquad (20)$$

The normalised certainty-equivalent discount function is given by:

$$e^{-r_{can}(t-s)} = \frac{\text{E}e^{-\tilde{r}(t-\tau)}}{\text{E}e^{-\tilde{r}(s-\tau)}} \qquad (21)$$

As such, the normalised certainty-equivalent average discount rate is:

$$r_{can}(t, s, \tau) = -\frac{1}{t-s} \ln \left[\frac{\text{E}e^{-\tilde{r}(t-\tau)}}{\text{E}e^{-\tilde{r}(s-\tau)}} \right] \qquad (22)$$

Note that the certainty-equivalent *marginal* discount rate does not depend upon the normalisation:

$$r_{cm}(t, \tau) = \frac{\text{E}\tilde{r}e^{-\tilde{r}(t-\tau)}}{\text{E}e^{-\tilde{r}(t-\tau)}} \qquad (23)$$

In order to establish that the certainty equivalent *average* discount rate is declining with t and increasing with τ, we employ the following two lemmas.

Lemma 4. *The certainty-equivalent marginal discount rate, $r_{cm}(t, \tau)$, is weakly montonically declining with time t and weakly monotonically increasing with evaluation date τ.*

Proof. Differentiating equation 23 shows that:

$$\frac{\partial r_{cm}}{\partial t} = \left(\frac{\mathrm{E}\tilde{r}e^{-\tilde{r}(t-\tau)}}{\mathrm{E}e^{-\tilde{r}(t-\tau)}} \right)^2 - \frac{\mathrm{E}\tilde{r}^2 e^{-\tilde{r}(t-\tau)}}{\mathrm{E}e^{-\tilde{r}(t-\tau)}} \tag{24}$$

By Jensen's inequality, the right hand side is non-positive for all t, proving that r_{cm} is weakly declining with t. It follows that r_{cm} is weakly increasing with τ by observing that $\partial r_{cm}/\partial t = -\partial r_{cm}/\partial \tau$. ∎

LEMMA 5. *$Lim_{t \to s}\ r_{can}(t, s, \tau) = r_{cm}(s, \tau)$.*

Proof. Taking the limit of equation (22) using L'Hôpital's rule gives:

$$\lim_{t \to s} r_{can}(t, s, \tau) = \frac{\mathrm{E}\tilde{r}e^{-\tilde{r}(s-\tau)}}{\mathrm{E}e^{-\tilde{r}(s-\tau)}} = r_{cm}(s, \tau) \tag{25}$$

by equation (23). ∎

We can now prove that the certainty-equivalent average discount rate declines with the passage of time, t, irrespective of the normalisation date, s.

PROPOSITION 6. *The normalised certainty-equivalent average discount rate, $r_{can}(t, \tau)$, is weakly monotonically declining with the passage of time, t.*

Proof. Differentiating equation (22) with respect to t shows that:

$$\frac{\partial r_{can}}{\partial t} = \left(\frac{1}{t-s} \right) (r_{cm} - r_{can}) \tag{26}$$

We proceed by contradiction. Suppose that for $t > s$, $r_{can} < r_{cm}$. Then $\partial r_{can}/\partial t > 0$ by equation (26). But as $r_{cm} = r_{can}$ at $t = s$ (Lemma 5), and $\partial r_{cm}/\partial t \leq 0$ (Lemma 4), it follows that $r_{can} > r_{cm}$ for $t > s$, establishing a contradiction. Hence for $t > s$, $r_{can} > r_{cm}$ and $\partial r_{can}/\partial t \leq 0$ by equation (26). An analogous argument holds for $t < s$. ∎

It simply remains to establish that the certainty-equivalent average discount rate is increasing with the evaluation time, τ, irrespective of the normalisation date, s.

PROPOSITION 7. *The normalised certainty-equivalent average discount rate,* $r_{can}(t, \tau)$, *is weakly monotonically increasing with the evaluation date* τ.

Proof. Differentiating equation (22) with respect to τ shows that:

$$\frac{\partial r_{can}}{\partial t} = \left(\frac{1}{t-s}\right)(r_{cm}(s, \tau) - r_{cm}(t, \tau)) \tag{27}$$

For $t > s$, $r_{cm}(s, \tau) \geq r_{cm}(t, \tau)$ because $\partial r_{cm}/\partial t \leq 0$ (Lemma 4). Thus $\partial r_{can}/\partial \tau \geq 0$ by equation (27). An analogous argument holds for $t < s$. ∎

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Table I: Two thought experiments on decision criteria and risk allocation.

Criterion	Future bears risk	Present bears risk
ENPV	Safe deposit	Safe deposit
ENFV	Market	Market